Relation Connecting Various Elastic Constants

(i) **Relation connecting Y, K and σ** is drive in class

" $Y \equiv$ Young's Modulus
$K \equiv$ Bulk Modulus
$\sigma \equiv$ Poission Ratio "

$$\boxed{K = \frac{Y}{3(1-2\sigma)}} \quad \text{— } (R_1)$$

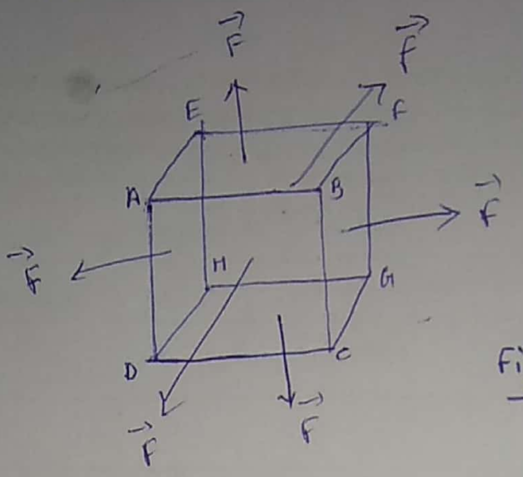

Fig-1

(ii) **Relation Connecting Y, η and σ**

" $Y \equiv$ Young's Modulus.
$\eta \equiv$ Modulus of Rigidity
$\sigma \equiv$ Poission Ratio

let ABCD represent the front face of a cube of side L. A tangential force F is applied on its upper face AB and the bottom face DC is fixed. As a result of this force the cube is sheared to A'B'CD through an angle θ. then, shearing strain is-

$$\tan\theta \simeq \theta = \frac{AA'}{AD} = \frac{BB'}{BC} = \frac{l}{L}$$

since θ is very small so $\tan\theta = \theta$
where displacement is $AA' = BB' = l$.

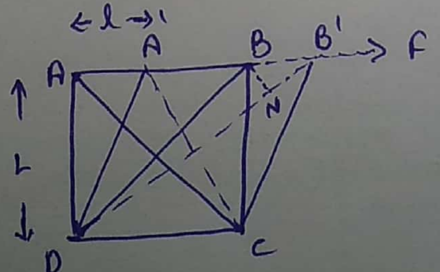

Fig-2

shearing stress $T = \dfrac{\text{tangential force}}{\text{Area of the upper face of the cube}}$

$$= \frac{F}{L^2}$$

$\therefore$ modulus of rigidity $\eta = \dfrac{\text{shearing stress}}{\text{shearing strain}}$

$$\eta = \frac{T}{\theta} = T \cdot \frac{L}{l} \quad - (1)$$

But as we show previous that shearing stress will equal to tensile and compressive stresses at right angle. so shearing stress along AB is equal to a tensile stress along DB and an equal compressive stress along AC at right angles to each other.

Let α and β be the longitudinal and lateral strains per unit stress respectively. Then extension along diagonal DB due to tensile stress -

$$= DB \cdot T \cdot \alpha.$$

and extension along diagonal DB due to compression stress along AC.

$$= DB \cdot T \cdot \beta.$$

Total extension along DB (NB') $= DB \cdot T \cdot (\alpha + \beta) = L\sqrt{2}\,T(\alpha+\beta).$

Let us draw a perpendicular BN on DB'. Then increase in the length of diagonal DB is practically equal to NB'. As θ is very small, therefore angle AB'C is nearly 90° and $\angle BB'N = \angle 45°$

Thus $NB' = BB' \cos 45° = \dfrac{BB'}{\sqrt{2}} = \dfrac{l}{\sqrt{2}}$ in $\Delta BNB'$

$\therefore$ $L\sqrt{2}\,T(\alpha+\beta) = \dfrac{l}{\sqrt{2}}$ or $\boxed{T\dfrac{L}{l} = \dfrac{1}{2(\alpha+\beta)}}$

From Eq (1)

$$\eta = \frac{1}{2(\alpha+\beta)}$$

or $$\eta = \frac{1}{2\alpha(1+\beta/\alpha)} \quad -(2)$$

Here $$\frac{\beta}{\alpha} = \frac{\text{Lateral strain}}{\text{longitudinal strain}} = \text{Poission Ratio } (\sigma)$$

and α is longitudinal strain per unit stress.

i.e. since ~~Yong~~ Young Modulus $Y = \frac{\text{stress}}{\text{longitudinal strain}}$

$$\Rightarrow \quad Y = \frac{1}{\alpha}$$

Therefore Eq (2) becomes

$$\eta = \frac{Y}{2(1+\sigma)}$$

or $$\boxed{Y = 2\eta(1+\sigma)} \quad -(R_2)$$

(iii) Relation connecting Y, K and η.

from relation R_1 & R_2

$$1-2\sigma = Y/3K$$

$$2+2\sigma = Y/\eta$$

Adding the above we get

$$3 = \frac{Y}{3K} + \frac{Y}{\eta}$$

$$3 = Y\left(\frac{1}{3K}+\frac{1}{\eta}\right) = Y\left(\frac{\eta+3K}{\eta 3K}\right)$$

$\Rightarrow$

$$9K\eta = Y(\eta + 3K)$$

$$\frac{9}{Y} = \frac{\eta + 3K}{K\eta}$$

$$\boxed{\frac{9}{Y} = \frac{1}{K} + \frac{3}{\eta}} \quad - (R_3).$$

(iv) Relation connecting K, η and σ

From relations R_1 & R_2 we have

$$Y = 3K(1-2\sigma) \quad \text{and} \quad Y = 2\eta(1+\sigma).$$

equating these two values of Y.

$$3K(1-2\sigma) = 2\eta(1+\sigma).$$

or $$3K - 6K\sigma = 2\eta + 2\eta\sigma$$

$$3K - 2\eta = 2\eta\sigma + 6K\sigma$$

$$3K - 2\eta = \cancel{2\eta}\ 2\sigma(\eta + 3K)$$

i.e $$\boxed{\sigma = \frac{3K - 2\eta}{2\eta + 6K}}$$

The limiting values of σ.

As above, we have seen.

$$3K(1-2\sigma) = 2\eta(1+\sigma).$$

where K and η are essentially positive quantities.

Therefore.

(i) If Poisson's ratio is a positive quantity ! then both right hand side and left hand side of above equation must be positive. This imply

$$3K(1-2\sigma) > 0$$

since K is > 0.

or $$(1-2\sigma) > 0$$

$(1-2\sigma) > 0.$

$1-2\sigma-1 > 0-1$

$-2\sigma > -1$

$\frac{2\sigma}{2} < \frac{1}{2}$

$\sigma < \frac{1}{2}$ or $\boxed{\sigma < 0.5}$

(ii) If σ is a negative quantity Then left hand side of equation $3K(1-2\sigma) = 2\eta(1+\sigma)$ will positive. Then the right hand side expression must also be positive. Hence

$1+\sigma > 0$

or

$1+\sigma-1 > 0-1$

$\boxed{\sigma > -1}$

Thus, the theoritical limiting values of σ are -1 and 0.5

i.e $\boxed{-1 < \sigma < 0.5}$